Evaluation of the Third Heat Kernel Coefficient in a Dirac Field Theory by Mathematica

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Abstract

A computational method by Mathematica developed by the author is extended to evaluate the third coefficient of the heat kernel for the Dirac field in the gravitational background field. Eighth-order derivative of the geodesic parallel displacement biscalar and sixth-order derivative of the geodesic bispinor in the coincidence limit are evaluated and their indices are contracted. The result of the third coefficient partly agrees with Gilkey's one, but there remaims some discrepancy. The reason is not traceable at present.
Keywords:heat kernel, trace anomaly, chiral anomaly, graviational anomaly, Mathematica

I Introduction

The chiral anomaly, the trace anomaly of the energy momentum tensor, and the gravitational anomaly can be derived from the heat kernel coefficient. In n space-time dimensions, the coefficient $a_{n/2}(x,x)$ is relevant. In the previous papers[1,2), we have presented the method to evaluate the first and second coefficients, a_1 and a_2 in scalar, spinor, and gauge field theories. They are relevant to the various anomalies in two and four space-time dimensions. There are huge numbers of papers to try to explain the exiting particles by the superstring theory[3)]. The superstring lives in ten dimensional space-time, and if the superstring theory is the fundamental theory, we need to know the coefficients up to fifth order. Towards this purpose, we extend the previous method to evaluate the third coefficient of the heat kernel in the present paper. The gravitational anomaly also appears in six space-time dimensions.

The fermionic determinant to be considered is the following one,

$$e^{iW[E^\mu_a]} \equiv \int [d\psi\, d\overline{\psi}]\, e^{i\int d^n x\sqrt{-g}\,\overline{\psi}\, i\not{D}\psi}\,, \tag{1.1}$$

and we regularize the determinant (1.1) as the previous paper:

$$W[E^\mu_a] = \frac{i}{2}\,\mathrm{Tr}\left[\int_\epsilon^\infty \frac{dt}{t}\, e^{-tY^{(1/2)}}\right], \tag{1.2}$$

with

$$Y^{(1/2)} = \not{D}^2 = g^{\mu\nu} D_\mu D_\nu + \xi R\,. \tag{1.3}$$

where ξ is $\frac{1}{4}$ in the Dirac case, but we retain ξ as a free parameter for the sake of generality.

The variation of the effective action (1.2) under an infinitesimal local Weyl transformation $E^\mu_a \to E^\mu_a(1+\alpha(x))$ yields the expression of the trace of the energy-momentum tensor as

$$\begin{aligned}&\sqrt{-g}\, g^{\mu\nu} < T_{\mu\nu}(x) > \\ &= -i \lim_{x'\to x} \mathrm{tr}\left[e^{-\epsilon\not{D}^2} I(x,x')\delta^{(n)}(x,x')\right].\end{aligned} \tag{1.4}$$

We follow the notation of the previous papers[1)]. $I(x,x')$ is the geodesic parallel displacement bispinor and its definition and the coincidence limits of its lower derivatives are given in the second paper of Ref.1.

The classical action of the massless Dirac theory is invariant under an infinitesimal chiral transformation, $\psi(x) \to (1+i\gamma_{n+1}\alpha)\psi(x)$. The introduction of the cutoff parameter with a mass-square dimension breaks this symmetry, and it yields the chiral anomaly. We introduce an external field A_μ and consider the following effective action,

$$W[E^\mu_a, A^\mu] = \frac{i}{2}\,\mathrm{Tr}\left[\int_\epsilon^\infty \frac{dt}{t}\, e^{-t\not{D}_A^2}\right], \tag{1.5}$$

where $\not{D}_A$ is defined by

$$\not{D}_A = \not{D} - i\gamma_\mu\gamma_{n+1}A^\mu\,. \tag{1.6}$$

By applying the gauge transformation $A^\mu \to A^\mu + \partial^\mu\alpha$ and setting the external field to zero, we obtain the following relation,

$$\begin{aligned}&W[E^\mu_a, \partial^\mu\alpha] - W[E^\mu_a, 0]\\ &= \frac{i}{2}\,\mathrm{Tr}\left[\int_\epsilon^\infty dt\,\{\not{D}, i\not{\partial}\alpha\gamma_{n+1}\}\, e^{-t\not{D}^2}\right]\\ &= -\frac{1}{2}\,\mathrm{Tr}\left[\int_\epsilon^\infty dt\,\{\not{D},\{\not{D},\alpha\gamma_{n+1}\}\}\, e^{-t\not{D}^2}\right]\\ &= -2\,\mathrm{Tr}\left[\alpha\gamma_{n+1}\, e^{-\epsilon\not{D}^2}\right].\end{aligned} \tag{1.7}$$

On the other hand, the variation of the effective action under the gauge transformation is related to the divergence of the axial vector current as follows:

$$\begin{aligned}&W[E^\mu_a, \partial^\mu\alpha] - W[E^\mu_a, 0]\\ &= \int d^n x\sqrt{-g}\,\partial^\mu\alpha\, \tfrac{\delta W}{\delta A^\mu}|_{A=0}\\ &= \int d^n x\sqrt{-g}\,\partial^\mu\alpha\, < \overline{\psi}\gamma_\mu\gamma_{n+1}\psi >\\ &= -\int d^n x\sqrt{-g}\,\alpha\,\nabla^\mu < \overline{\psi}\gamma_\mu\gamma_{n+1}\psi >\,.\end{aligned} \tag{1.8}$$

Comparing Eqs.(1.7) and (1.8), we obtain the chiral anomaly relation as follows:

$$\begin{aligned}&\sqrt{-g}\,\nabla_\mu < \overline{\psi}\gamma_\mu\gamma_{n+1}\psi > \\ &= 2\lim_{x'\to x}\mathrm{tr}\big[\gamma_{n+1}\,e^{-\epsilon \not{D}^2} I(x,x')\delta^{(n)}(x,x')\big]\,. \quad (1.9)\end{aligned}$$

The expression for the chiral anomaly (1.9) is also derivable from the path integral measure[4].

If we consider a chiral Dirac field theory in which only the left-handed fermion is involved, the conservation law of the energy-momentum tensor is violated in six space-time dimensions, or generally in $4k+2\,(k:\text{integer})$ space-time dimensions. Formally the right-handed fermion is introduced, but it does not couple to the gravitational field. The fermionic determinant to be considered is modified as

$$e^{iW[E^\mu_a]} \equiv \int [d\psi\, d\overline{\psi}]\, e^{i\int d^nx\sqrt{-g}\,\overline{\psi}\, i\mathcal{D}\,\psi}\,, \qquad (1.10)$$

with

$$\begin{aligned}\mathcal{D} &= \not{D}\frac{1-\gamma_{n+1}}{2} + \not{\partial}\frac{1+\gamma_{n+1}}{2} \\ &= \not{D}P_- + \not{\partial}P_+\,. \qquad (1.11)\end{aligned}$$

Here $\not{\partial} = \delta^\mu_a\gamma^a\partial_\mu$, and $P_-(P_+)$ is the projection operator to the left(right)-handed fermion. Then $Y^{(1/2)}$ and its exponentiated expression are modified as follows:

$$\begin{aligned}Y^{(1/2)} &= \not{\partial}\not{D}P_- + \not{D}\not{\partial}P_+\,, \qquad (1.12)\\ e^{-tY^{(1/2)}} &= e^{-t\not{\partial}\not{D}}P_- + e^{-t\not{D}\not{\partial}}P_+\,. \qquad (1.13)\end{aligned}$$

The classical action is invariant under the so-called covariant transformation[5] $E^\mu_a \to E^\mu_a - (\nabla_\nu\xi^\mu)E^\nu_{\ a}$, which is a special combination of the local Lorentz transformation and the general coordinate transformation. This classical invariance is also violated by the introduction of the cutoff parameter ϵ, that is,

$$\begin{aligned}&W[E^\mu_a - (\nabla_\nu\xi^\mu)E^\nu_{\ a}] - W[E^\mu_a]\\ &= -\frac{i}{2}\mathrm{Tr}\Big[\int_\epsilon^\infty dt\,\{\not{\partial}\,[\not{D},\,\xi^\mu D_\mu]\,e^{-t\not{\partial}\not{D}}P_- \\ &\qquad + [\not{D},\,\xi^\mu D_\mu]\,\not{\partial}\,e^{-t\not{D}\not{\partial}}P_+\}\Big]\\ &= -i\,\mathrm{Tr}\Big[\int_\epsilon^\infty dt\,\{\xi^\mu D_\mu\not{\partial}\not{D}e^{-t\not{\partial}\not{D}}P_- \\ &\qquad - \xi^\mu D_\mu\not{D}\not{\partial}\,e^{-t\not{D}\not{\partial}}P_+\}\Big]\\ &= -i\,\mathrm{Tr}\Big[\xi^\mu D_\mu\,\{e^{-\epsilon\not{\partial}\not{D}}P_- - e^{-\epsilon\not{D}\not{\partial}}P_+\}\Big]\,. \qquad (1.14)\end{aligned}$$

Meanwhile, the variation of the effective action under the covariant transformation, is related to the divergence of the energy-momentum tensor as follows:

$$\begin{aligned}&W[E^\mu_a - (\nabla_\nu\xi^\mu)E^\nu_{\ a}] - W[E^\mu_a]\\ &= -\int d^nx\,(\nabla_\nu\xi^\mu)E^\nu_{\ a}\frac{\delta W}{\delta E^\mu_a}\\ &= -\int d^nx\sqrt{-g}\,(\nabla^\nu\xi^\mu) < T_{\mu\nu}(x) > \\ &= \int d^nx\sqrt{-g}\,\xi^\mu(x)\nabla^\nu < T_{\mu\nu}(x) > \,. \qquad (1.15)\end{aligned}$$

Comparing Eqs.(1.14) and (1.15), we obtain the garvitational anomaly as follows:

$$\begin{aligned}&\sqrt{-g}\,\nabla^\nu < T_{\mu\nu}(x) > = -i\lim_{x'\to x}\mathrm{tr}\Big[D_\mu\,\{e^{-\epsilon\not{\partial}\not{D}}P_- \\ &\quad -e^{-\epsilon\not{D}\not{\partial}}P_+\}I(x,x')\delta^{(n)}(x,x')\Big]\,. \qquad (1.16)\end{aligned}$$

This is the so-called consistent anomaly and it differs to the so-called covariant anomaly by local polynomials of the spin connection and its derivatives[6]. The covariant anomaly is related to the heat kernel coefficient by

$$\begin{aligned}&\sqrt{-g}\,\nabla^\nu < T_{\mu\nu}(x) >_{\rm cov}\\ &= i\lim_{x'\to x}\mathrm{tr}\big[\gamma_{n+1}D_\mu\, e^{-\epsilon\not{D}^2}\, I(x,x')\,\delta^{(n)}(x,x')\big]\,. \quad (1.17)\end{aligned}$$

As explained in the previous paper[1], we use the following covariant expression for the delta function,

$$\delta^{(n)}(x,x') = \int\frac{d^nk}{(2\pi)^n}e^{ik_a\sigma^a(x,x')}\,. \qquad (1.18)$$

II Perturbative expansion

As in the previous papers[1], we first commute the factor $e^{ik\cdot\sigma}$ with $Y^{(1/2)}$,

$$\begin{aligned}&Y^{(1/2)}\,e^{ik_a\sigma^a}\\ &= e^{ik_a\sigma^a}\{g^{\mu\nu}(-i\kappa_\mu + D_\mu)(-i\kappa_\nu + D_\nu) + \xi R\}\\ &= e^{ik_a\sigma^a}\big(K - iX^{(1/2)} + Y^{(1/2)}\big)\,, \qquad (2.1)\end{aligned}$$

with

$$\begin{aligned}\kappa_\mu &= -k_a\sigma_\mu^{\ a}\,, \qquad (2.2)\\ K &= -\kappa^2 \qquad (2.3)\\ X^{(1/2)} &= \kappa^\rho D_\rho + D_\rho\kappa^\rho\,. \qquad (2.4)\end{aligned}$$

Exponentiating the relation (2.1), we have

$$e^{-tY^{(1/2)}}e^{ik_a\sigma^a} = e^{ik_a\sigma^a}e^{-t(K-iX^{(1/2)}+Y^{(1/2)})}\,. \quad (2.5)$$

By regarding K as the free Hamiltonian, we apply the perturbation theory,

$$\begin{aligned}&e^{-t(K+V)} = e^{-tK}\\ &+\iint_0^t dt_1\,dt_2\,\delta(t-t_1-t_2)\,e^{-t_1K}\,V\,e^{-t_2K}\\ &+\iiint_0^t dt_1\,dt_2\,dt_3\,\delta(t-\sum_{i=1}^3 t_i)e^{-t_1K}Ve^{-t_2K}Ve^{-t_3K}\\ &+\cdots\,. \qquad (2.6)\end{aligned}$$

As in the previous paper[1], we move the covariant derivative D_μ to the rightmost position at first. Then we can take the coincidence limit of e^{-tK}, and it reduces to e^{tk^2}.

Making use of the following formula, we can manage the commutator of the covariant derivative D_μ with e^{-tK} as a perturbation:

$$\begin{aligned}
&\int\cdots\int_0^t \cdots dt_i \cdots \delta(t-\cdots-t_i-\cdots) \\
&\times\cdots[D_\mu,\, e^{-t_i K}]\cdots \\
&= -\int\cdots\int_0^t \cdots dt_i \cdots \delta(t-\cdots-t_i-\cdots)\int\!\!\int_0^{t_i} dt_i'\, dt_i'' \\
&\times\delta(t_i-t_i'-t_i'')\cdots e^{-t_i' K}[D_\mu,\, K]\, e^{-t_i'' K}\cdots \\
&= -\int\cdots\int_0^t \cdots dt_i'\, dt_i'' \cdots \delta(t-\cdots-t_i'-t_i''-\cdots) \\
&\times\cdots e^{-t_i' K}[D_\mu,\, K]\, e^{-t_i'' K}\cdots.
\end{aligned} \tag{2.7}$$

The program to move the covariant derivatives to the rightmost position is almost the same as the previous ones .

We denote the contribution to the heat kernel coefficient from the m-ple X and n-ple Y insertions as $\mathcal{A}^{(mn)}$. We have to evaluate $\mathcal{A}^{(mn)}$'s, with $(m,n) = (6,0), (4,1), (2,2), (0,3)$ in six space-time dimensions. Taking into account the fact that X and Y involve the covariant derivative once and twice respectively, we have terms consisted of sixth derivatives of I, K, κ, fourth derivative of R, or products of lower derivatives of I, K, κ, R, after moving the covariant derivatives involved in X and Y to the rightmost position,

The integrations over the momentum and the parameters are performed with the help of the following formula,

$$\begin{aligned}
&\int\prod_{i=1}^{j}\delta(\epsilon-\sum_{i=1}^{j}t_i)\int\frac{d^nk}{(2\pi)^n}e^{\epsilon k^2}k_{\mu_1}k_{\mu_2}\cdots k_{\mu_{2m}} \\
&= \frac{\epsilon^{(j-1)}\,\Gamma(\omega)}{2^m(j-1)!\,\Gamma(\omega+m)}\, g_{\mu_1\mu_2\cdots\mu_{2m}}\int\frac{d^nk}{(2\pi)^n}e^{\epsilon k^2}(k^2)^m \\
&= \frac{i\sqrt{-g}}{(4\pi)^\omega}\frac{(-1)^m}{2^m(j-1)!}\, g_{\mu_1\mu_2\cdots\mu_{2m}}\,\epsilon^{j-1-\omega-m},
\end{aligned} \tag{2.8}$$

where $\omega = \frac{n}{2}$ and the generalized metric g was introduced in the previous paper[5,1]. It is defined by the recursive relation,

$$g_{\mu_1\mu_2\cdots\mu_{2m}} = \sum_{j=2}^{2m} g_{\mu_1\mu_j}\, g_{\mu_2\cdots\check{\mu}_j\cdots\mu_{2m}} \quad (m\geq 2)\,, \tag{2.9}$$

where $\check{\mu}_j$ means that the index μ_j is eliminated from the indices of g.

The factor $\Gamma(\omega)/\Gamma(\omega+m)$ appeared after averaging over the angle variables of the momentum, which we call the symmetrization operation, eventually disappears after the momentum integration. We count only the factor $1/2^m$ by the symmetrization operation, and other factors are taken into account at the final step of summing up all terms. In our calculation of the heat kernel coefficient, j and m are related to each other, that is,

$$j-1-\omega-m=0\,. \tag{2.10}$$

III Computation of $I_{\mu_1\cdots\mu_8}$ and $\sigma_{\mu_1\cdots\mu_7}{}^a$, and their contractions

To compute the heat kernel coefficients, we need to calculate $I_{\mu_1\cdots\mu_8}$ and $\sigma_{\mu_1\cdots\mu_7}{}^a$. The fundamental recursion equations are as follows:

$$\sigma^\mu I_\mu = 0\,, \tag{3.1}$$

$$\sigma_\mu = \sigma_{\mu\nu}\,\sigma^\nu\,, \tag{3.2}$$

$$\sigma^a = \sigma_\mu{}^a\sigma^\mu\,, \tag{3.3}$$

$$\begin{aligned}
&[D_\mu,\, D_\nu]\, I_{\rho_1\cdots\rho_m} \\
&= \hat{R}_{\mu\nu}\, I_{\rho_1\cdots\rho_m} - \sum_{j=1}^{m} R_{\mu\nu}{}^{\tau}{}_{\rho_j} I_{\rho_1\cdots\tau\cdots\rho_m}\,,
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&[\nabla_\mu,\, \nabla_\nu]\,\sigma_{\rho_1\cdots\rho_m} \\
&= -\sum_{j=1}^{m} R_{\mu\nu}{}^{\tau}{}_{\rho_j}\sigma_{\rho_1\cdots\tau\cdots\rho_m}\,,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&[\nabla_\mu,\, \nabla_\nu]\,\sigma_{\rho_1\cdots\rho_m}{}^a \\
&= -\sum_{j=1}^{m} R_{\mu\nu}{}^{\tau}{}_{\rho_j}\sigma_{\rho_1\cdots\tau\cdots\rho_m}{}^a\,.
\end{aligned} \tag{3.6}$$

The program to solve the recursion equations and to calculate contractions of their indices are presented in Appendices A, B, and C.

We denote $[\sigma^a_{\mu_1\cdots\mu_m}]$ and $[I_{\mu_1\cdots\mu_m}]$ as $s[\mu_1,\cdots,\mu_m,a]$ and $i[\mu_1,\cdots,\mu_m]$ in the computer programs, respectively. If their indices are completely symmetrized, they vanish. In fact, $s[\mu_1,\cdots,\mu_m]$ vanishes, if all indices except one are symmetrized. $i[\mu_1,\cdots,\mu_m]$ vanishes, if all indices except the last one are symmetrized. Then, $g[\mu_1,\cdots\mu_8]*s[\mu_1,\cdots,\mu_8]=0$, $g[\mu_1,\cdots\mu_6]*i[\mu_1,\cdots,\mu_6]=0$ and so on. Furthermore, $g[\mu_1,\cdots,\mu_{12}]*s[\mu_1,\cdots,\mu_4]*s[\mu_5,\cdots,\mu_8]*s[\mu_9,\cdots,\mu_{12}]=0$, $g[\mu_1,\cdots,\mu_{10}]*s[\mu_1,\cdots,\mu_4]*s[\mu_5,\cdots,\mu_8]*s[\mu_9,\mu_{10},\nu,\nu]=0$, $g[\mu_1,\cdots,\mu_{10}]*s[\mu_1,\cdots,\mu_5]*s[\mu_6,\cdots,\mu_{10}]=0$ and so on. The terms proportional to the generalized metric with eight indices are not survival. Only surviving terms proportional to the generalized metric with six indices are $g[\mu_1,\cdots,\mu_6]*s[\mu_1,\mu_2,\cdots]*s[\mu_3,\mu_4,\cdots]*s[\mu_5,\mu_6,\cdots]$, $g[\mu_1,\cdots,\mu_6]*s[\mu_1,\mu_2,\cdots]*s[\mu_3,\cdots,\mu_6,\cdots]$, $g[\mu_1,\cdots,\mu_6]*s[\mu_1,\mu_2,\mu_3,\cdots]*s[\mu_4,\mu_5,\mu_6,\cdots]$, and so on.

IV Results

In six space-time dimensions the third coefficient of the heat kernel is obtained as follows:

$$\begin{aligned}
&\sum_{(m,n)=\{(6,0),(4,1),(2,2),(0,3)\}}\mathcal{A}^{(mn)} \\
&= \frac{1}{64\pi^3}\,\mathrm{tr}\big[c_1\,\Box^2 R + c_2\, R\,\Box R + c_3\, R^{\mu\nu}\,\Box R_{\mu\nu} \\
&\quad + c_4\, R^{\mu\nu}\,\nabla_\mu\nabla_\nu R + c_5\, R^{\mu\nu\rho\sigma}\,\Box R_{\mu\nu\rho\sigma} \\
&\quad + c_6\,(\nabla_\mu R)^2 + c_7\,(\nabla_\mu R_{\nu\rho})^2 + c_8\,(\nabla^\mu R^{\nu\rho})\,\nabla_\nu R_{\mu\rho} \\
&\quad + c_9\,(\nabla_\mu R_{\nu\rho\sigma\tau})^2 + c_{10}\, R^3 + c_{11}\, R\, R_{\mu\nu}^2 \\
&\quad + c_{12}\, R\, R_{\mu\nu\rho\sigma}^2 + c_{13}\, R^\mu{}_\nu\, R^\nu{}_\rho\, R^\rho{}_\mu
\end{aligned}$$

$$
\begin{aligned}
&+c_{14}\, R_{\mu\rho}\, R_{\nu\sigma}\, R^{\mu\nu\rho\sigma} + c_{15}\, R^{\mu}{}_{\nu}\, R_{\mu\rho\sigma\tau}\, R^{\nu\rho\sigma\tau} \\
&+c_{16}\, R_{\mu\nu\rho\sigma}\, R^{\mu\nu}{}_{\tau\lambda}\, R^{\rho\sigma\tau\lambda} + c_{17}\, R_{\mu\rho\nu\sigma}\, R^{\mu}{}_{\tau}{}^{\nu}{}_{\lambda}\, R^{\rho\tau\sigma\lambda} \\
&+d_1\,(D^2 \hat{R}_{\mu\nu})\, \hat{R}^{\mu\nu} + d_2\, \hat{R}^{\mu\nu}\, D^2 \hat{R}_{\mu\nu} \\
&+d_3\, R^{\mu\nu}\, D_\mu D^\rho \hat{R}_{\nu\rho} + d_4\,(D_\mu \hat{R}_{\nu\rho})^2 + d_5\,(D^\mu \hat{R}_{\mu\nu})^2 \\
&+d_6\,(\nabla^\mu R)\, D^\nu \hat{R}_{\mu\nu} + d_7\,(\nabla_\mu R^{\mu\nu\rho\sigma})\, D_\nu \hat{R}_{\rho\sigma} \\
&+d_8\, R\, \hat{R}^2_{\mu\nu} + d_9\, R^{\mu}{}_{\nu}\, \hat{R}_{\mu\rho}\, \hat{R}^{\nu\rho} \\
&+d_{10}\, R^{\mu\nu\rho\sigma}\, \hat{R}_{\mu\nu}\, \hat{R}_{\rho\sigma} + d_{11}\, \hat{R}^{\mu}{}_{\nu}\, \hat{R}_{\mu\rho}\, \hat{R}^{\nu\rho} \big] . \qquad (4.1)
\end{aligned}
$$

Other tensor structures are reducible to the ones in Eq.(4.1) with the help of the Bianchi identities. For example,

$$R^{\mu\rho\nu\sigma}\Box R_{\mu\nu\rho\sigma} = \frac{1}{2}\, R^{\mu\nu\rho\sigma}\Box R_{\mu\nu\rho\sigma}\,, \qquad (4.2)$$

$$R^{\mu\nu\rho\sigma}\nabla_\tau \nabla_\mu R_{\tau\nu\rho\sigma} = \frac{1}{2}\, R^{\mu\nu\rho\sigma}\Box R_{\mu\nu\rho\sigma}\,, \qquad (4.3)$$

$$
\begin{aligned}
&R^{\mu\nu\rho\sigma}\nabla_\mu \nabla_\tau R_{\tau\nu\rho\sigma} = 2R^{\mu\nu\rho\sigma}\nabla_\mu \nabla_\sigma R_{\nu\rho} = \\
&\quad \frac{1}{2}\, R^{\mu\nu\rho\sigma}\Box R_{\mu\nu\rho\sigma} + R^{\mu}{}_{\nu} R_{\mu\alpha\beta\gamma} R^{\nu\alpha\beta\gamma} + \frac{1}{2}\, R_{\mu\nu\rho\sigma} \\
&\quad \times R^{\mu\nu}{}_{\tau\lambda} R^{\rho\sigma\tau\lambda} + 2R_{\mu\rho\nu\sigma} R^{\mu}{}_{\tau}{}^{\nu}{}_{\lambda} R^{\rho\tau\sigma\lambda}\,, \qquad (4.4)
\end{aligned}
$$

$$
\begin{aligned}
&R^{\mu\nu}\nabla^\rho \nabla_\mu R_{\nu\rho} = \frac{1}{2}\, R^{\mu\nu}\nabla_\mu \nabla_\nu R \\
&\quad -R^{\mu}{}_{\nu} R^{\nu}{}_{\rho} R^{\rho}{}_{\mu} - R_{\mu\nu} R_{\rho\sigma} R^{\mu\nu\rho\sigma}\,, \qquad (4.5)
\end{aligned}
$$

$$\hat{R}^{\mu\nu} D^\rho D_\mu \hat{R}_{\nu\rho} = -\frac{1}{2}\, \hat{R}^{\mu\nu} D^2 \hat{R}_{\mu\nu}\,, \qquad (4.6)$$

$$
\begin{aligned}
&\hat{R}^{\mu\nu} D_\mu D^\rho \hat{R}_{\nu\rho} = -\frac{1}{2}\, \hat{R}^{\mu\nu} D^2 \hat{R}_{\mu\nu} + 2\hat{R}^{\mu}{}_{\nu} \hat{R}_{\mu\rho} \hat{R}^{\nu\rho} \\
&\quad -\frac{1}{2}\, R^{\mu\nu\rho\sigma} \hat{R}_{\mu\nu} \hat{R}_{\rho\sigma} - R^{\mu}{}_{\nu} \hat{R}_{\mu\rho} \hat{R}^{\nu\rho}\,, \qquad (4.7)
\end{aligned}
$$

$$(\nabla^\mu R^{\nu\rho}) D_\nu \hat{R}_{\mu\rho} = \frac{1}{2}\, (\nabla_\mu R^{\mu\nu\rho\sigma}) D_\nu \hat{R}_{\rho\sigma}\,, \qquad (4.8)$$

$$(D^\mu \hat{R}^{\nu\rho}) D_\nu \hat{R}_{\mu\rho} = \frac{1}{2}\, (D_\mu \hat{R}_{\nu\rho})^2\,, \qquad (4.9)$$

$$R^{\mu\rho\nu\sigma} \hat{R}_{\mu\nu} \hat{R}_{\rho\sigma} = \frac{1}{2}\, R^{\mu\nu\rho\sigma} \hat{R}_{\mu\nu} \hat{R}_{\rho\sigma}\,. \qquad (4.10)$$

The coefficients c's and d's are shown in Table 1 and 2, respectively. These results agree with those by Gilkey[8,9,10], except fot the coefficients c_{13}, c_{14}, d_8, and d_{11}. At present, we are not able to trace the origin of the discrepancy.

If we take the trace of the third coefficient of the heat kernel with respect to the Dirac indices, we obtain the explicit form of the trace anomaly, the ciral anomaly, and the gravitational anomaly in six space-time dimensions. The trace calculation will be left for the future publication, together with the identification of the reason for the discrepancy between our result and Gilkey's one.

Appendix A. Recursive solution of $I_{\mu_1\cdots\mu_6}$

```
SetDirectory["d:\\trace_anomaly2\contract"]

SetAttributes[Pd,Flat];
(* i[w] stands for I, w is a dummy,
  z involves I.  rs[\mu,\nu,\cdots,\rho,\tau]
  stands for D_\mu D_\nu \hat{R}_{\rho\tau}.
```

tensor (symbol)	c's value (general ξ)	$\xi = \frac{1}{4}$
$\Box^2 R$ (ddddsc)	$-\frac{1}{280} + \frac{\xi}{60}$	$\frac{1}{1680}$
$R\,\Box R$ (sc*ddsc)	$\frac{1}{180} - \frac{11\xi}{180} + \frac{\xi^2}{6}$	$\frac{1}{1440}$
$R^{\mu\nu}\,\Box R_{\mu\nu}$ (rddr1)	$-\frac{1}{630}$	$-\frac{1}{630}$
$R^{\mu\nu}\,\nabla_\mu \nabla_\nu R$ (rddsc)	$\frac{1}{420} - \frac{\xi}{90}$	$-\frac{1}{2520}$
$R^{\mu\nu\rho\sigma}\,\Box R_{\mu\nu\rho\sigma}$ (cddc)	$\frac{1}{420}$	$\frac{1}{420}$
$(\nabla_\mu R)^2$ (dscsq)	$\frac{17}{5040} - \frac{\xi}{30} + \frac{\xi^2}{12}$	$\frac{1}{4032}$
$(\nabla_\mu R_{\nu\rho})^2$ (drsq2)	$-\frac{1}{2520}$	$-\frac{1}{2520}$
$(\nabla^\mu R^{\nu\rho})\,\nabla_\nu R_{\mu\rho}$ (drsq3)	$-\frac{1}{1260}$	$-\frac{1}{1260}$
$(\nabla_\mu R_{\nu\rho\sigma\tau})^2$ (dcsq)	$\frac{1}{560}$	$\frac{1}{560}$
R^3 (sc^3)	$-\frac{1}{1296} + \frac{\xi}{72} - \frac{\xi^2}{12} + \frac{\xi^3}{6}$	$\frac{1}{10368}$
$R\,R^2_{\mu\nu}$ (sc*rsq)	$\frac{1}{1080} - \frac{\xi}{180}$	$-\frac{1}{2160}$
$R\,R^2_{\mu\nu\rho\sigma}$ (sc*csq)	$-\frac{1}{1080} + \frac{\xi}{180}$	$\frac{1}{2160}$
$R^{\mu}{}_{\nu}\,R^{\nu}{}_{\rho}\,R^{\rho}{}_{\mu}$ (rcu)	$-\frac{1}{5670}$	$-\frac{1}{5670}$
$R_{\mu\rho}\,R_{\nu\sigma}\,R^{\mu\nu\rho\sigma}$ (crsq)	$-\frac{1}{1890}$	$-\frac{1}{1890}$
$R^{\mu}{}_{\nu}\,R_{\mu\rho\sigma\tau}\,R^{\nu\rho\sigma\tau}$ (rcsq)	$\frac{1}{945}$	$\frac{1}{945}$
$R_{\mu\nu\rho\sigma}\,R^{\mu\nu}{}_{\tau\lambda}\,R^{\rho\sigma\tau\lambda}$ (ccu1)	$\frac{11}{11340}$	$\frac{11}{11340}$
$R_{\mu\rho\nu\sigma}\,R^{\mu}{}_{\tau}{}^{\nu}{}_{\lambda}\,R^{\rho\tau\sigma\lambda}$ (ccu2)	$\frac{1}{567}$	$\frac{1}{567}$

Table 1: The third coefficient of the heat kernel(c's)

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  \sigma_{\mu\nu\cdots\rho} is denoted by
  s0[\mu,\nu,\cdots,\rho]. *)
Pd[x___,y1_+y2_,z___]:=
  Pd[x,y1,z] + Pd[x,y2,z];
Pd[x___,0,y___]:= 0
Pd[x___,c_*rs[y__],z___]:= c*Pd[x,rs[y],z];
Pd[x___,c_*Pd[y__],z___]:= c*Pd[x,y,z];
Pd[x___,d[m_],rs[y__],z___]:=
  Pd[x,rs[y],d[m],z] + Pd[x,rs[m,y],z];
Pd[x___,d[m_],s0[y__],z___]:=
```

tensor (symbol)	d's value	tensor (symbol)	d's value
$(D^2\hat{R}_{\mu\nu})\,\hat{R}^{\mu\nu}$ (spcddspc1)	$\frac{1}{60}$	$(\nabla_\mu R^{\mu\nu\rho\sigma})\,D_\nu\hat{R}_{\rho\sigma}$ (dcdspc)	0
$\hat{R}^{\mu\nu}\,D^2\hat{R}_{\mu\nu}$ (spcddspc2)	$\frac{1}{60}$	$R\,\hat{R}^2_{\mu\nu}$ (sc*spcsq)	$-\frac{1}{48}$
$R^{\mu\nu}\,D_\mu D^\rho\hat{R}_{\nu\rho}$ (rddspc)	0	$R^\mu{}_\nu\,\hat{R}_{\mu\rho}\,\hat{R}^{\nu\rho}$ (rspcsq)	$\frac{1}{90}$
$(D_\mu\hat{R}_{\nu\rho})^2$ (dspcsq1)	$\frac{1}{45}$	$R^{\mu\nu\rho\sigma}\,\hat{R}_{\mu\nu}\,\hat{R}_{\rho\sigma}$ (cspcsq)	$\frac{1}{60}$
$(D^\mu\hat{R}_{\mu\nu})^2$ (dspcsq2)	$\frac{1}{180}$	$\hat{R}^\mu{}_\nu\,\hat{R}_{\mu\rho}\,\hat{R}^{\nu\rho}$ (spccu)	$-\frac{1}{60}$
$(\nabla^\mu R)\,D^\nu\hat{R}_{\mu\nu}$ (dscdspc)	0		

Table 2: The third coefficient of the heat kernel(d's)

```
  Pd[x,s0[y],d[m],z] + Pd[x,s0[m,y],z];
Pd[x___,d[m_],r[y__],z___]:=
  Pd[x,r[y],d[m],z] + Pd[x,r[m,y],z];
Pd[x__,d[m_],i[y__]]:= Pd[x,i[m,y]];
Pd[x__,d[m_]]:= 0;

(* commutation relations for i[].  n is a
  fixed index.  i[m1,m2,\cdots,m6,w] is
  specifically denoted by
  it[m1,m2,\cdots,m6] *)
it[k__,l2_,l1_]:= it[k,l1,l2] -
  Apply[Pd,Join[Thread[d[{k}]],
  {rs[l1,l2],i[w]}]]/;  OrderedQ[{l1,l2}];
it[k__,l2_,l1_,m__]:= it[k,l1,l2,m] -
  Apply[Pd,Join[Thread[d[{k}]],
  {rs[l1,l2],i[m,w]}]] - Sum[
  Apply[Pd,Join[Thread[d[{k}]],
  {r[l1,l2,{m}[[j]],n],
  Apply[i,ReplacePart[{m,w},n,j]]}]],
  {j,1,Length[{m}]}]/; OrderedQ[{l1,l2}];
it[l2_,l1_,m__]:= it[l1,l2,m] -
  Pd[rs[l1,l2],i[m,w]] - Sum[
  Pd[r[l1,l2,{m}[[j]],n],
  Apply[i,ReplacePart[{m,w},n,j]]],
  {j,1,Length[{m}]}]/; OrderedQ[{l1,l2}];

counter=1
(* Deivative of Eq.(3.1) is identically
  zero.  n is a fixed index. *)
f[counter]=
  Pd[d[m1],d[m2],d[m3],d[m4],d[m5],d[m6],
  s0[n],i[n,w]]//.
  {Pd[s0[y_,n],i[x__,n,w]]->it[x,y]};
(* rs[m1,m2,m3,m4,m5,m6] is specifically
  denoted by rst[m1,m2,m3,m4,m5,m6] *)
f[counter+1]=f[counter]//. {
  Pd[rs[k__],i[w]]->rst[k]};
counter=counter+1

(* recursive definition of it[m1,\cdots,m6]
   The second term is identically zero.   *)
f[counter+1]=it[m1,m2,m3,m4,m5,m6] -
  Expand[(f[counter]//. rs[m_]->0)/6];
counter=counter+1

f[counter+1]=f[counter]//. {
  Pd[x___,r[y__],z___]->r[y]*Pd[x,z],
  Pd[x___,s0[y__],z___]->s0[y]*Pd[x,z],
  Pd[rs[x__]]->rs[x],
  Pd[i[x__]]->i[x]};
counter=counter+1

(* %%%% The following part until the end
  mark is commonly used in B and C programs.
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% *)
(* inclusion of definitions of lower
  derivatives of I and s *)
Get["i4s6form.txt"]

(* sc stands for the scalar curvature, R. *)
r[m_,n_,n_,m_]:= sc;
r[m_,n_,m_,n_]:= -sc;
r[m_,m_,x_,y_]:= 0;
r[x__,m_,m_]:= 0;
r[x_,m_,m_,y_,z_]:= 0;

(* ri[\mu,\nu] stands for the Ricci tensor,
  R_{\mu\nu}. *)
r[m_,x_,y_,m_]:= Apply[ri,Sort[{x,y}]];
r[x_,m_,m_,y_]:= Apply[ri,Sort[{x,y}]];
r[m_,x_,m_,y_]:= -Apply[ri,Sort[{x,y}]];
r[x_,m_,y_,m_]:= -Apply[ri,Sort[{x,y}]];

r[x_,m_,y_,z_,m_]:=
  Apply[ri,Join[{x},Sort[{y,z}]]];
r[x_,y_,m_,m_,z_]:=
  Apply[ri,Join[{x},Sort[{y,z}]]];
r[x_,y_,m_,z_,m_]:=
  -Apply[ri,Join[{x},Sort[{y,z}]]];
r[x_,m_,y_,m_,z_]:=
  -Apply[ri,Join[{x},Sort[{y,z}]]];
(* %%%% the end of the common part %%%% *)

rs[x___,m_,m_]:= 0;
rs[x___,m_,n_,m_,n_]:= 0;
rs[x___,m_,n_,n_,m_]:= 0;
f[counter+1]=Expand[f[counter]];
counter=counter+1

i6=f[counter]//. {
  Pd[rs[x__]]->rs[x],s0[x_,y_,z_]->0};

Save["defi6_r.txt",i6]
```

Appendix B. Contraction of indices of $I_{\mu_1\cdots\mu_6}$

```
SetDirectory["d:\\trace_anomaly2\contract"]

Get["defi6_r.txt"]

SetAttributes[Pd,Flat];
Pd[x___,y1_+y2_,z___]:=
  Pd[x,y1,z] + Pd[x,y2,z];
Pd[x___,0,y___]:= 0
Pd[x___,c_*rs[y__],z___]:= c*Pd[x,rs[y],z];
Pd[x___,d[m_],rs[y__],z___]:=
  Pd[x,rs[y],d[m],z] + Pd[x,rs[m,y],z];
Pd[x___,d[m_],r[y__],z___]:=
  Pd[x,r[y],d[m],z] + Pd[x,r[m,y],z];
Pd[x__,d[m_]]:= 0;

counter=1
(* definition of contracted i(6)
i[m_,m_,n_,n_,o_,o_,m7]:= i6c1;
i[m_,m_,n_,o_,o_,n_,m7]:= i6c2;
i[m_,n_,n_,m_,o_,o_,m7]:= i6c3;
i[m_,n_,n_,o_,o_,m_,m7]:= i6c4; *)
h[1,counter]=i6//.
  {m1->l1,m2->l1,m3->l2,m4->l2,m5->l3,m6->l3};
h[2,counter]=i6//.
  {m1->l1,m2->l1,m3->l2,m4->l3,m5->l3,m6->l2};
h[3,counter]=i6//.
  {m1->l1,m2->l2,m3->l2,m4->l1,m5->l3,m6->l3};
h[4,counter]=i6//.
  {m1->l1,m2->l2,m3->l2,m4->l3,m5->l3,m6->l1};

rst[k__,m_,m_]:= 0;
rst[k__,m_,n_,m_,n_]:= 0;
rst[k__,m_,n_,n_,m_]:= 0;

(* reordering of indices of rst[] *)
rst[m_,x_,m_,y__]:= rst[m,m,x,y] -
  Pd[d[m],rs[m,x],rs[y]] +
  Pd[d[m],rs[y],rs[m,x]] -
  Sum[Apply[Pd,Join[{d[m]},{r[m,x,{y}[[j]],n],
  Apply[rs,ReplacePart[{y},n,j]]}]],
  {j,1,3}];
rst[x_,m_,m_,y__]:= rst[m,x,m,y] -
  Pd[rs[m,x],rs[m,y]] + Pd[rs[m,y],rs[m,x]] -
  Sum[Pd[r[m,x,{m,y}[[j]],n],
  Apply[rs,ReplacePart[{m,y},n,j]]],{j,1,4}];
rst[x_,m_,y_,m_,z__]:= rst[x,m,m,y,z] -
  Pd[d[x],d[m],rs[m,y],rs[z]] +
  Pd[d[x],d[m],rs[z],rs[m,y]] -
  Sum[Apply[Pd,Join[{d[x],d[m]},
  {r[m,y,{z}[[j]],n],Apply[rs,
  ReplacePart[{z},n,j]]}]],{j,1,2}];
rst[x_,y_,m_,m_,z__]:= rst[x,m,y,m,z] -
  Pd[d[x],rs[m,y],rs[m,z]] +
  Pd[d[x],rs[m,z],rs[m,y]] -
  Sum[Apply[Pd,Join[{d[x]},
  {r[m,y,{m,z}[[j]],n], Apply[rs,
  ReplacePart[{m,z},n,j]]}]], {j,1,3}];

Do[h[j,counter+1]=Expand[h[j,counter]],{j,1,4}]
counter=counter+1

Do[h[j,counter+1]=h[j,counter]//. {
  Pd[x___,r[y__],z___]->r[y]*Pd[x,z],
  Pd[rs[x__]]->rs[x]},{j,1,4}]
counter=counter+1

(* %%%%% Here the common part mentioned in
  Appendix A is inserted. %%%%%%%%%%%%%% *)

rs[x___,m_,m_]:= 0;
rs[m_,n_,m_,n_]:= 0;
rs[m_,n_,n_,m_]:= 0;

(* reordering of last two indices of rs[]
  Reordered rs[] is denoted by ors[]. *)
rs[z___,x_,y_]:= ors[z,x,y]/; OrderedQ[{x,y}];
rs[z___,y_,x_]:= -ors[z,x,y]/; OrderedQ[{x,y}];
Pd[x___,-ors[y__],z___]:= - Pd[x,ors[y],z];

(* reduction of various tensor structures *)
subst1="{ri[x_,y_]*ors[z__,x_,y_]->0,
(* rs(2)^3 *)
  Pd[ors[x_,y_],ors[x_,z_],ors[y_,z_]]
    -> spccu,
  Pd[ors[x_,z_],ors[x_,y_],ors[y_,z_]]
    -> -spccu,
  Pd[ors[x_,y_],ors[y_,z_],ors[x_,z_]]
    -> -spccu,
  Pd[ors[x_,z_],ors[y_,z_],ors[x_,y_]]
    -> spccu,
  Pd[ors[y_,z_],ors[x_,y_],ors[x_,z_]]
    -> spccu,
  Pd[ors[y_,z_],ors[x_,z_],ors[x_,y_]]
    -> -spccu,
(* rs(3)^2 *)
(* The arguments of the 1st or[] are
   x_,y_,z_. *)
  Pd[ors[],ors[x_,y_,z_]]-> dspcsq1,
  Pd[ors[],ors[x_,z_,y_]]-> -dspcsq1,
  Pd[ors[],ors[y_,x_,z_]]-> dspcsq1/2,
  Pd[ors[],ors[y_,z_,x_]]-> -dspcsq1/2,
  Pd[ors[],ors[z_,x_,y_]]-> -dspcsq1/2,
  Pd[ors[],ors[z_,y_,x_]]-> dspcsq1/2,

  Pd[ors[m_,m_,x_],ors[n_,n_,x_]]->dspcsq2,
  Pd[ors[m_,x_,m_],ors[n_,n_,x_]]->-dspcsq2,
  Pd[ors[m_,m_,x_],ors[n_,x_,n_]]->-dspcsq2,
  Pd[ors[m_,x_,m_],ors[n_,x_,n_]]->dspcsq2,
(* rs(2)*rs(4) *)
(* The arguments of the 2nd ors[] are x_,y_. *)
  Pd[ors[m_,m_,x_,y_],ors[]]-> spcddspc1,
  Pd[ors[m_,x_,m_,y_],ors[]]-> spcddspc1/2,
  Pd[ors[m_,x_,y_,m_],ors[]]-> -spcddspc1/2,
  Pd[ors[x_,m_,m_,y_],ors[]]->
    spcddspc1/2-spccu+rspcsq+cspcsq/2,
  Pd[ors[x_,m_,y_,m_],ors[]]->
```

```
    -spcddspc1/2+spccu-rspcsq-cspcsq/2,
(* Similar expressions where x and y in the
  second ors[] are exchanged follow. *)

(* The arguments of the 1st ors[] are x_,y_. *)
  Pd[ors[],ors[m_,m_,x_,y_]]-> spcddspc2,
  Pd[ors[],ors[m_,x_,m_,y_]]-> spcddspc2/2,
  Pd[ors[],ors[m_,x_,y_,m_]]-> -spcddspc2/2,
  Pd[ors[],ors[x_,m_,m_,y_]]->
   spcddspc2/2-spccu+rspcsq+cspcsq/2,
  Pd[ors[],ors[x_,m_,y_,m_]]->
    -spcddspc2/2+spccu-rspcsq-cspcsq/2,
(* Similar expressions where x and y in the
  first ors[] are exchanged follow. *)

(* rs(2)^2*ricci *)
  Pd[ors[x_,y1_],ors[x_,y2_]]*ri[z1_,z2_]
    -> rspcsq,
  Pd[ors[x_,y1_],ors[y2_,x_]]*ri[z1_,z2_]
    -> -rspcsq,
  Pd[ors[y1_,x_],ors[x_,y2_]]*ri[z1_,z2_]
    -> -rspcsq,
  Pd[ors[y1_,x_],ors[y2_,x_]]*ri[z1_,z2_]
    -> rspcsq,
(* rs(2)^2*r(4) *)
  Pd[ors[x_,y_],ors[z_,w_]]*r[u1_,u2_,
    u3_,w_]-> cspcsq[x,y,z,u1,u2,u3],
  Pd[ors[x_,y_],ors[z_,w_]]*r[u1_,u2_,
    w_,u3_]-> -cspcsq[x,y,z,u1,u2,u3],
  Pd[ors[x_,y_],ors[z_,w_]]*r[u1_,w_,
    u2_,u3_]-> cspcsq[x,y,z,u2,u3,u1],
  Pd[ors[x_,y_],ors[z_,w_]]*r[w_,u1_,
    u2_,u3_]-> -cspcsq[x,y,z,u2,u3,u1],

  r[n_,n_,x_,y_,z_]->-dc[y,z,x],
  r[n_,x_,n_,y_,z_]->dc[y,z,x],
  r[n_,x_,y_,n_,z_]->-dc[x,y,z],
  r[n_,x_,y_,z_,n_]->dc[x,y,z]}";

Do[h[j,counter+1]=Expand[h[j,counter]]//.
  ToExpression[subst1],{j,1,4}]
counter=counter+1

subst2="{r[x__]*r[y__]*ors[z1_,z2_]->0,
  r[x__]*ri[y__]*ors[z1_,z2_]->0,
  ri[x__]*ri[y__]*ors[z1_,z2_]->0,
  r[x1_,x2_,x3_,x4_,x5_,x6_]*ors[y1_,y2_]->0,
(* r(4)*rs(4) *)
  r[x1_,x2_,x3_,x4_]*ors[y1_,y2_,y3_,y4_]->0,
  ri[x_,y_]*ors[x_,m_,y_,m_]-> rddspc,
  ri[x_,y_]*ors[x_,m_,m_,y_]-> -rddspc,
  ri[x_,y_]*ors[m_,x_,m_,y_]-> -rddspc,
  ri[x_,y_]*ors[m_,x_,y_,m_]-> rddspc,
(* Similar expressions where x and y in ri[]
  are exchanged follow. *)
(* r(5)*rs(3) *)
  ri[m_,n_,n_]*ors[k_,k_,m_]-> -dscdspc,
  ri[m_,n_,n_]*ors[k_,m_,k_]-> dscdspc,
  ri[n_,m_,n_]*ors[k_,k_,m_]-> -dscdspc/2,
  ri[n_,m_,n_]*ors[k_,m_,k_]-> dscdspc/2,
  ri[n_,n_,m_]*ors[k_,k_,m_]-> -dscdspc/2,
  ri[n_,n_,m_]*ors[k_,m_,k_]-> dscdspc/2,
(* The arguments of the 2nd ors[] are
  m1_,m2_,m3_. *)
  dc[m1_,m2_,m3_]*ors[]-> dcdspc/2,
  dc[m1_,m3_,m2_]*ors[]-> -dcdspc/2,
  dc[m2_,m1_,m3_]*ors[]-> -dcdspc/2,
  dc[m2_,m3_,m1_]*ors[]-> -dcdspc,
  dc[m3_,m1_,m2_]*ors[]-> dcdspc/2,
  dc[m3_,m2_,m1_]*ors[]-> dcdspc,

  ri[m1_,m2_,m3_]*ors[]-> 0,
  ri[m1_,m3_,m2_]*ors[]-> 0,
  ri[m2_,m1_,m3_]*ors[]-> -dcdspc/2,
  ri[m2_,m3_,m1_]*ors[]-> -dcdspc/2,
  ri[m3_,m2_,m1_]*ors[]-> dcdspc/2,
  ri[m3_,m1_,m2_]*ors[]-> dcdspc/2,
(* rs(2)^2*r(4) *)
  cspcsq[x_,y_,z_,x_,y_,z_]->cspcsq,
  cspcsq[x_,y_,z_,x_,z_,y_]->cspcsq/2,
  cspcsq[x_,y_,z_,y_,x_,z_]->-cspcsq,
  cspcsq[x_,y_,z_,y_,z_,x_]->-cspcsq/2,
  cspcsq[x_,y_,z_,z_,x_,y_]->-cspcsq/2,
  cspcsq[x_,y_,z_,z_,y_,x_]->cspcsq/2}";

Do[h[j,counter+1]=Expand[h[j,counter]]//.
  ToExpression[subst2],{j,1,4}]
counter=counter+1

i6c1= Expand[h[1,counter]]
i6c2= Expand[h[2,counter]]
i6c3= Expand[h[3,counter]]
i6c4= Expand[h[4,counter]]

Save["i6cont_r.txt",i6c1,i6c2,i6c3,i6c4]
```

Appendix C. Recursive solution of $\sigma_{\mu_1 \cdots \mu_7}{}^a$ and contraction of its indices

```
SetDirectory["d:\\trace_anomaly2\contract"]

(* s[\mu,\nu,\cdots,\rho] stands for \sigma_
  {\mu,\nu,\cdots}^\rho. *)
SetAttributes[Pd,Flat];
Pd[x___,y1_+y2_,z___]:=
  Pd[x,y1,z] + Pd[x,y2,z];
Pd[x___,c_*y_,z___]:= c*Pd[x,y,z]
  /; NumberQ[c];
Pd[x___,d[m_],s0[y__],z___]:=
  Pd[x,s0[y],d[m],z] + Pd[x,s0[m,y],z];
Pd[x___,d[m_],s[y__],z___]:=
  Pd[x,s[y],d[m],z] + Pd[x,s[m,y],z];
Pd[x___,d[m_],r[y__],z___]:=
  Pd[x,r[y],d[m],z] + Pd[x,r[m,y],z];
Pd[x__,d[m_]]:= 0;
```

```
counter=1
(* deivative of Eq.(3.3) is identically zero.
  n is a fixed index. *)
f[counter]=st[m1,m2,m3,m4,m5,m6,m7,p]-
  Pd[d[m1],d[m2],d[m3],d[m4],d[m5],d[m6],
  d[m7],s[n,p],s0[n]]//. {
  Pd[s[x__,n,p],s0[y_,n]]->st[x,y,p],
  Pd[s[n,p],s0[x__,n]]->-s0[x,p]};

(* reordering of indices in s(8),
  n and p are fixed indices.
  s[m1,m2,\cdots,m7,p] is specifically
  denoted by st[m1,\cdots,p].          *)
st[k__,m2_,m1_,p]:= st[k,m1,m2,p]/;
  OrderedQ[{m1,m2}];
st[m2_,m1_,x__,p]:= st[m1,m2,x,p] -
  Sum[Pd[r[m1,m2,{x}[[j]],n],
  Apply[s,Join[ReplacePart[{x},n,j],{p}]]],
  {j,1,Length[{x}]}]/; OrderedQ[{m1,m2}];
st[k__,m2_,m1_,x__,p]:= st[k,m1,m2,x,p] -
  Sum[Apply[Pd,Join[Thread[d[{k}]],
  {r[m1,m2,{x}[[j]],n],Apply[s,
  Join[ReplacePart[{x},n,j],{p}]]}]],
  {j,1,Length[{x}]}]/; OrderedQ[{m1,m2}];

f[counter+1]=f[counter];
counter=counter+1

f[counter+1]=f[counter]//. Pd->Times;
counter=counter+1

(* recursive definition of st[m1,\cdots,m7,p]
   The second term is identically zero.    *)
f[counter+1]=Expand[st[m1,m2,m3,m4,
  m5,m6,m7,p]+f[counter]/6]//. {
  s0[m_]->0,s0[m_,n_]->g[m,n],s0[m_,n_,o_]->0,
  s[m_]->0,s[m_,n_]->-g[m,n],s[m_,n_,o_]->0};
counter=counter+1

f[counter+1]=f[counter]//. {
  g[x_,y_]*s0[z1___,x_,z2___]->s0[z1,y,z2],
  g[x_,y_]*s[z1___,x_,z2___]->s[z1,y,z2],
  g[x_,y_]*r[z1___,x_,z2___]->r[z1,y,z2]};
counter=counter+1

(* r(5) *)
r[x_,m_,n_,m_,n_]:= -sc[x];
r[x_,m_,n_,n_,m_]:= sc[x];
r[m_,x_,n_,m_,n_]:= -sc[x]/2;
r[m_,x_,n_,n_,m_]:= sc[x]/2;
r[m_,n_,x_,m_,n_]:= sc[x]/2;
r[m_,n_,x_,n_,m_]:= -sc[x]/2;
r[m_,m_,n_,x_,n_]:= -sc[x]/2;
r[m_,n_,m_,x_,n_]:= sc[x]/2;
r[m_,m_,n_,n_,x_]:= sc[x]/2;
r[m_,n_,m_,n_,x_]:= -sc[x]/2;

(* %%%% Here the common part mentioned in
  Appendix A is inserted. %%%%%%%%%%%%%% *)

(* Bianchi identity *)
r[m_,m_,x_,y_,z_]:=
  -Apply[ri,Join[{y},Sort[{x,z}]]]+
  Apply[ri,Join[{z},Sort[{x,y}]]];
r[m_,x_,m_,y_,z_]:=
  Apply[ri,Join[{y},Sort[{x,z}]]]-
  Apply[ri,Join[{z},Sort[{x,y}]]];
r[m_,x_,y_,m_,z_]:=
  -Apply[ri,Join[{x},Sort[{y,z}]]]+
  Apply[ri,Join[{y},Sort[{x,z}]]];
r[m_,x_,y_,z_,m_]:=
  Apply[ri,Join[{x},Sort[{y,z}]]]-
  Apply[ri,Join[{y},Sort[{x,z}]]];
(* r(6) *)
r[x_,y_,m_,n_,m_,n_]:= -Apply[sc,Sort[{x,y}]];
r[x_,y_,m_,n_,n_,m_]:=  Apply[sc,Sort[{x,y}]];

sc[m_,m_]:= ddsc;
ri[x_,m_,m_]:= sc[x];
ri[m_,m_,x_]:= sc[x]/2;
ri[m_,x_,m_]:= sc[x]/2;

r[x_,y_,m_,z_,m_,w_]:=
  -Apply[ri,Join[{x,y},Sort[{z,w}]]];
r[x_,y_,z_,m_,w_,m_]:=
  -Apply[ri,Join[{x,y},Sort[{z,w}]]];
r[x_,y_,z_,m_,m_,w_]:=
  Apply[ri,Join[{x,y},Sort[{z,w}]]];
r[x_,y_,m_,z_,w_,m_]:=
  Apply[ri,Join[{x,y},Sort[{z,w}]]];
ri[m_,n_,m_,n_]:= ddsc/2;

s8=Expand[f[counter]];
Save["defs8_r.txt",s8]

(* application of Bianchi identity *)
r[x_,m_,m_,y_,z_,w_]:=
  Apply[ri,Join[{x,w},Sort[{y,z}]]]-
  Apply[ri,Join[{x,z},Sort[{y,w}]]];
r[x_,m_,y_,m_,z_,w_]:=
  -Apply[ri,Join[{x,w},Sort[{y,z}]]]+
  Apply[ri,Join[{x,z},Sort[{y,w}]]];
r[x_,m_,y_,z_,m_,w_]:=
  -Apply[ri,Join[{x,y},Sort[{z,w}]]]+
  Apply[ri,Join[{x,z},Sort[{y,w}]]];
r[x_,m_,y_,z_,w_,m_]:=
  Apply[ri,Join[{x,y},Sort[{z,w}]]]-
  Apply[ri,Join[{x,z},Sort[{y,w}]]];

ri[m_,m_,n_,n_]:= ddsc;
ri[m_,n_,m_,n_]:= ddsc/2;
ri[m_,n_,n_,m_]:= ddsc/2;

ri[x_,m_,m_,y_]:= Apply[sc,Sort[{x,y}]]/2;
ri[x_,m_,y_,m_]:= Apply[sc,Sort[{x,y}]]/2;
ri[m_,x_,m_,y_]:= Apply[sc,Sort[{x,y}]]/2 -
```

```
  ri[x,q1]*ri[y,q1] + ri[m,q1]*r[m,x,y,q1];
ri[m_,x_,y_,m_]:= Apply[sc,Sort[{x,y}]]/2 -
  ri[x,q1]*ri[y,q1] + ri[m,q1]*r[m,x,y,q1];

r[x_,y_,z_,w_,u_,v_,m_,m_]:= 0;
r[x_,y_,z_,w_,m_,m_,u_,v_]:= 0;
r[x_,y_,z_,w_,m_,n_,n_,m_]:=
  Apply[sc,Join[{x,y},Sort[{z,w}]]];
r[x_,y_,z_,w_,m_,n_,m_,n_]:=
  -Apply[sc,Join[{x,y},Sort[{z,w}]]];

(* rddr2 stands for R_{\mu\nu}
  \nabla_\rho\nabla^\mu R^{\rho\nu}, and it
  is eliminated at the final step.        *)
sc[m_,m_,n_,n_]:= ddddsc;
sc[m_,n_,m_,n_]:= ddddsc - dscsq/2 -2*rddr2 -
  2*rcu -2*crsq;
sc[m_,n_,n_,m_]:= ddddsc - dscsq/2 -2*rddr2 -
  2*rcu -2*crsq;

sc[x_,y_,m_,m_]:= ddsc[x,y];
sc[x_,m_,y_,m_]:= ddsc[x,y] -ri[x,y,q2]*
  sc[q2] -ri[y,q2]*sc[x,q2];
sc[x_,m_,m_,y_]:= ddsc[x,y] -ri[x,y,q2]*
  sc[q2] -ri[y,q2]*sc[x,q2];

ddsc[m_,m_]:= ddddsc;

r[x_,y_,z_,w_,m_,u_,v_,m_]:=
  Apply[ri,Join[{x,y,z,w},Sort[{u,v}]]];
r[x_,y_,z_,w_,u_,m_,m_,v_]:=
  Apply[ri,Join[{x,y,z,w},Sort[{u,v}]]];
r[x_,y_,z_,w_,m_,u_,m_,v_]:=
  -Apply[ri,Join[{x,y,z,w},Sort[{u,v}]]];
r[x_,y_,z_,w_,u_,m_,v_,m_]:=
  -Apply[ri,Join[{x,y,z,w},Sort[{u,v}]]];
r[x_,y_,z_,m_,m_,w_,u_,v_]:=
  Apply[ri,Join[{x,y,z,v},Sort[{u,w}]]] -
  Apply[ri,Join[{x,y,z,u},Sort[{v,w}]]];
r[x_,y_,z_,m_,w_,m_,u_,v_]:=
  -Apply[ri,Join[{x,y,z,v},Sort[{u,w}]]] +
  Apply[ri,Join[{x,y,z,u},Sort[{v,w}]]];
r[x_,y_,z_,m_,w_,u_,m_,v_]:=
  -Apply[ri,Join[{x,y,z,w},Sort[{u,v}]]] +
  Apply[ri,Join[{x,y,z,u},Sort[{v,w}]]];
r[x_,y_,z_,m_,w_,u_,v_,m_]:=
  Apply[ri,Join[{x,y,z,w},Sort[{u,v}]]]  -
  Apply[ri,Join[{x,y,z,u},Sort[{v,w}]]];

(* ordering in ri, q is a fixed index. *)
ri[x_,y_,z_,w_,m_,m_]:=
  Apply[sc,Join[{x,y},Sort[{z,w}]]];
ri[x_,y_,z_,m_,w_,m_]:=
  Apply[sc,Join[{x,y},Sort[{z,w}]]]/2;
ri[x_,y_,z_,m_,m_,w_]:=
  Apply[sc,Join[{x,y},Sort[{z,w}]]]/2;
ri[x_,y_,m_,z_,m_,w_]:=
  Apply[sc,Join[{x,y},Sort[{z,w}]]]/2
  - ri[x,y,z,q]*ri[w,q] - ri[x,z,q]*ri[y,w,q]
  - ri[y,z,q]*ri[x,w,q] - ri[z,q]*ri[x,y,w,q]
  + r[x,y,m,z,w,q]*ri[m,q] + r[x,m,z,w,q]*
  ri[y,m,q] + r[y,m,z,w,q]*ri[x,m,q]
  + r[m,z,w,q]*ri[x,y,m,q];
ri[x_,y_,m_,z_,w_,m_]:=
  Apply[sc,Join[{x,y},Sort[{z,w}]]]/2
  - ri[x,y,z,q]*ri[w,q] - ri[x,z,q]*ri[y,w,q]
  - ri[y,z,q]*ri[x,w,q] - ri[z,q]*ri[x,y,w,q]
  + r[x,y,m,z,w,q]*ri[m,q] + r[x,m,z,w,q]*
  ri[y,m,q] + r[y,m,z,w,q]*ri[x,m,q]
  + r[m,z,w,q]*ri[x,y,m,q];
ri[x_,y_,m_,m_,z_,w_]:= ri[x,m,y,m,z,w]
  + ri[x,y,q]*ri[q,z,w] + ri[y,q]*ri[x,q,z,w]
  + r[x,y,m,z,q]*ri[m,w,q] + r[y,m,z,q]*
  ri[x,m,w,q] + r[x,y,m,w,q]*ri[m,z,q]
  + r[y,m,w,q]*ri[x,m,z,q];

counter=1
(* definition of contracted s(8)
s[m_,m_,n_,n_,o_,o_,p_,p_]:= so1;
s[m_,m_,n_,n_,p_,o_,o_,p_]:= so2;
s[m_,m_,p_,n_,n_,o_,o_,p_]:= so3;
s[p_,m_,m_,n_,n_,o_,o_,p_]:= so4; *)
h[1,counter]=Expand[s8//.
  {m1->l1,m2->l1,m3->l2,m4->l2,
  m5->l3,m6->l3,m7->l4,p->l4}];
h[2,counter]=Expand[s8//.
  {m1->l1,m2->l1,m3->l2,m4->l2,
  m5->l4,m6->l3,m7->l3,p->l4}];
h[3,counter]=Expand[s8//.
  {m1->l1,m2->l1,m3->l4,m4->l2,
  m5->l2,m6->l3,m7->l3,p->l4}];
h[4,counter]=Expand[s8//.
  {m1->l4,m2->l1,m3->l1,m4->l2,
  m5->l2,m6->l3,m7->l3,p->l4}];

(* inclusion of definition of s0(8) *)
Get["defs08_r.txt"]
(* inclusion of expressions of contracted
  s0(8)(s08co1,\cdot,s08co4) *)
Get["s08cont_r.txt"]

Do[h[j,counter+1]=Expand[h[j,counter]],{j,1,4}]
counter=counter+1

subst1="{
  s0[l1,l1,l2,l2,l3,l3,l4,l4]-> s08co1,
  s0[l1,l1,l2,l2,l4,l3,l3,l4]-> s08co2,
  s0[l1,l1,l4,l2,l2,l3,l3,l4]-> s08co3,
  s0[l4,l1,l1,l2,l2,l3,l3,l4]-> s08co4,

(* r(1)^2, r(3)^2  *)
  sc[x_]^2->dscsq,ri[x_,y_,z_]^2-> drsq2,
  ri[x_,y_,z_]*ri[x_,z_,y_]-> drsq2,
  ri[x_,x2_,x3_]*ri[y1_,x_,y3_]-> drsq3,
  ri[x_,x2_,x3_]*ri[y1_,y2_,x_]-> drsq3,
(* r(2)^2r,r(2)^3,r(2)^2r(4) *)
```

```
  ri[x_,y_]^2->rsq,
  ri[x_,y_]*r[u__,x_,y_,z1_,z2_]->0,
  ri[y_,x_]*r[u__,x_,y_,z1_,z2_]->0,
  ri[x_,y_]*r[u__,x_,y_]->0,
  ri[y_,x_]*r[u__,x_,y_]->0,
  ri[x_,y_]*ri[x_,z_]*ri[y_,z_]-> rcu,
(* The arguments of the 1st and 2nd ri[] are
   x_,y_ and z_,w_, respectively. *)
  ri[]*ri[]*r[x_,z_,y_,w_]-> crsq,
  ri[]*ri[]*r[x_,w_,y_,z_]-> crsq,
  ri[]*ri[]*r[y_,z_,x_,w_]-> crsq,
  ri[]*ri[]*r[y_,w_,x_,z_]-> crsq,
  ri[]*ri[]*r[x_,z_,y_,w_]-> crsq,
  ri[]*ri[]*r[x_,z_,w_,y_]-> -crsq,
  ri[]*ri[]*r[x_,w_,z_,y_]-> -crsq,
  ri[]*ri[]*r[y_,z_,w_,x_]-> -crsq,
  ri[]*ri[]*r[y_,w_,z_,x_]-> -crsq}";
Do[h[j,counter+1]=h[j,counter]//.
  ToExpression[subst1],{j,1,4}]
counter=counter+1

(* r(2)r(6) *)
subst2="{
  ri[x_,y_]*sc[x_,y_]-> 2*rddr2+2*rcu+2*crsq,

  ri[x_,y_]*ri[m_,m_,x_,y_]-> rddr1,
  ri[x_,y_]*ri[m_,x_,m_,y_]-> rddr2,
  ri[x_,y_]*ri[m_,x_,y_,m_]-> rddr2,
  ri[x_,y_]*ri[x_,m_,m_,y_]-> rddr2+rcu+crsq,
  ri[x_,y_]*ri[x_,m_,y_,m_]-> rddr2+rcu+crsq,
(* Similar expressions where ri[x_,y_] is
   replaced by ri[y_,x_] follow. *)
(* The arguments of ri[] are x_,y_. *)
  ri[]*r[m_,n_,x_,m_,y_,n_]-> -rddr1+rddr2,
  ri[]*r[m_,n_,x_,m_,n_,y_]-> rddr1-rddr2,
  ri[]*r[m_,n_,m_,x_,y_,n_]-> rddr1-rddr2,
  ri[]*r[m_,n_,m_,x_,n_,y_]-> -rddr1+rddr2,
(* Similar expressions where ri[x_,y_] is
   replaced by ri[y_,x_] follow. *)
(* The arguments of ri[] are x_,y_. *)
  ri[]*r[m_,n_,x_,n_,y_,m_]-> -rddr1+rddr2,
  ri[]*r[m_,n_,x_,n_,m_,y_]-> rddr1-rddr2,
  ri[]*r[m_,n_,n_,x_,y_,m_]-> rddr1-rddr2,
  ri[]*r[m_,n_,n_,x_,m_,y_]-> -rddr1+rddr2,
(* Similar expressions where ri[x_,y_] is
   replaced by ri[y_,x_] follow. *)
  }";
Do[h[j,counter+1]=Expand[h[j,counter]//.
  ToExpression[subst2]],{j,1,4}]
counter=counter+1

(* r(4)r(6) *)
subst3="{
  r[n_,m1_,m2_,n_,m3_,m4_]->
    -r[n,m1,n,m2,m3,m4],
  r[n_,m1_,m2_,m3_,n_,m4_]->
    r[n,m1,n,m4,m2,m3],
  r[n_,m1_,m2_,m3_,m4_,n_]->
    -r[n,m1,n,m4,m2,m3],
  r[m1_,n_,m2_,n_,m3_,m4_]->
    -r[m1,n,n,m2,m3,m4],
  r[m1_,n_,m2_,m3_,n_,m4_]->
    r[m1,n,n,m4,m2,m3],
  r[m1_,n_,m2_,m3_,m4_,n_]->
    -r[m1,n,n,m4,m2,m3]}";
Do[h[j,counter+1]=h[j,counter]//.
  ToExpression[subst3],{j,1,4}]
counter=counter+1

subst4="{
  r[m_,x_,y_,w_]*r[n_,n_,z1_,z2_,z3_,m_]->
    -r[y,w,x,m]*r[n,n,z1,z2,z3,m],
  r[x_,m_,y_,w_]*r[n_,n_,z1_,z2_,z3_,m_]->
    r[y,w,x,m]*r[n,n,z1,z2,z3,m],
  r[x_,y_,m_,w_]*r[n_,n_,z1_,z2_,z3_,m_]->
    -r[x,y,w,m]*r[n,n,z1,z2,z3,m],

  r[m_,x_,y_,w_]*r[n_,z1_,n_,z2_,z3_,m_]->
    -r[y,w,x,m]*r[n,z1,n,z2,z3,m],
  r[x_,m_,y_,w_]*r[n_,z1_,n_,z2_,z3_,m_]->
    r[y,w,x,m]*r[n,z1,n,z2,z3,m],
  r[x_,y_,m_,w_]*r[n_,z1_,n_,z2_,z3_,m_]->
    -r[x,y,w,m]*r[n,z1,n,z2,z3,m],

  r[m_,x_,y_,w_]*r[z1_,n_,n_,z2_,z3_,m_]->
    -r[y,w,x,m]*r[z1,n,n,z2,z3,m],
  r[x_,m_,y_,w_]*r[z1_,n_,n_,z2_,z3_,m_]->
    r[y,w,x,m]*r[z1,n,n,z2,z3,m],
  r[x_,y_,m_,w_]*r[z1_,n_,n_,z2_,z3_,m_]->
    -r[x,y,w,m]*r[z1,n,n,z2,z3,m]}";
Do[h[j,counter+1]=h[j,counter]//.
ToExpression[subst4],{j,1,4}]
counter=counter+1

subst5="{
(* The arguments of the 2nd r[] are
   n_,n_,m1_,m2_,m3_,m4_. *)
  r[m1_,m2_,m3_,m4_]*r[]-> cddc,
  r[m1_,m3_,m2_,m4_]*r[]-> cddc/2,
  r[m2_,m1_,m3_,m4_]*r[]-> -cddc,
  r[m2_,m3_,m1_,m4_]*r[]-> -cddc/2,
  r[m3_,m1_,m2_,m4_]*r[]-> -cddc/2,
  r[m3_,m2_,m1_,m4_]*r[]-> cddc/2,

(* The arguments of the 2nd r[] are
   n_,m1_,n_,m2_,m3_,m4_. *)
  r[m1_,m2_,m3_,m4_]*r[]-> cddc/2,
  r[m1_,m3_,m2_,m4_]*r[]-> cddc/4,
  r[m2_,m1_,m3_,m4_]*r[]-> -cddc/2,
  r[m2_,m3_,m1_,m4_]*r[]-> -cddc/4,
  r[m3_,m1_,m2_,m4_]*r[]-> -cddc/4,
  r[m3_,m2_,m1_,m4_]*r[]-> cddc/4,

(* The arguments of the 2nd r[] are
   m1_,n_,n_,m2_,m3_,m4_. *)
  r[m1_,m2_,m3_,m4_]*r[]-> -2*aaa,
```

```
  r[m1_,m3_,m2_,m4_]*r[]-> -aaa,
  r[m2_,m1_,m3_,m4_]*r[]-> 2*aaa,
  r[m2_,m3_,m1_,m4_]*r[]-> aaa,
  r[m3_,m1_,m2_,m4_]*r[]-> aaa,
  r[m3_,m2_,m1_,m4_]*r[]-> -aaa}";
Do[h[j,counter+1]=h[j,counter]//.
  ToExpression[subst5],{j,1,4}]
counter=counter+1

(* r(2)r(4)^2 *)
subst6="
  ri[x_,y_]*r[z1___,x_,z2___]*r[z3___,y_,
    z4___]->ri*r[z1,k1,z2]*r[z3,k2,z4]";
Do[h[j,counter+1]=h[j,counter]//.
  ToExpression[subst6],{j,1,4}]
counter=counter+1

subst7="{
  r[x_,k1,y__]-> -r[k1,x,y],
  r[x__,k1,y_]-> r[k1,y,x],
  r[x__,y_,k1]-> -r[k1,y,x],
(* Similar expressions where k1 is replaced
   by k2 follow. *)              }";
Do[h[j,counter+1]=h[j,counter]//.
  ToExpression[subst7],{j,1,4}]
counter=counter+1

subst8="{
(* The arguments of the 1st r[] are
   x_,m1_,m2_,m3_. *)
  ri*r[]*r[y_,m1_,m2_,m3_]-> rcsq,
  ri*r[]*r[y_,m1_,m3_,m2_]-> -rcsq,
  ri*r[]*r[y_,m2_,m1_,m3_]-> rcsq/2,
  ri*r[]*r[y_,m2_,m3_,m1_]-> -rcsq/2,
  ri*r[]*r[y_,m3_,m1_,m2_]-> -rcsq/2,
  ri*r[]*r[y_,m3_,m2_,m1_]-> rcsq/2}";
Do[h[j,counter+1]=h[j,counter]//.
  ToExpression[subst8],{j,1,4}]
counter=counter+1

(* r(4)r(6) *)
subst9="{
  r[n_,m1_,m2_,m3_]*ri[m4__,n_]->
    -r[m2,m3,m1,n]*ri[m4,n],
  r[m1_,n_,m2_,m3_]*ri[m4__,n_]->
    r[m2,m3,m1,n]*ri[m4,n],
  r[m1_,m2_,n_,m3_]*ri[m4__,n_]->
    -r[m1,m2,m3,n]*ri[m4,n]}";
Do[h[j,counter+1]=h[j,counter]//.
  ToExpression[subst9],{j,1,4}]
counter=counter+1

subst10="{
(* The arguments of ri[] are
   m1_,m2_,m3_,n_. *)
  r[m1_,m2_,m3_,n_]*ri[]-> 0,
  r[m1_,m3_,m2_,n_]*ri[]-> aaa,
  r[m2_,m1_,m3_,n_]*ri[]-> 0,
  r[m2_,m3_,m1_,n_]*ri[]-> aaa,
  r[m3_,m1_,m2_,n_]*ri[]-> -aaa,
  r[m3_,m2_,m1_,n_]*ri[]-> -aaa}";
Do[h[j,counter+1]=h[j,counter]//.
  ToExpression[subst10],{j,1,4}]
counter=counter+1

r[x_,y_,z_,w_]:= or[x,y,z,w]/;
  SameQ[w,Sort[{x,y,z,w}][[4]]]
r[x_,y_,w_,z_]:= -or[x,y,z,w]/;
  SameQ[w,Sort[{x,y,z,w}][[4]]]
r[x_,w_,y_,z_]:= or[y,z,x,w]/;
  SameQ[w,Sort[{x,y,z,w}][[4]]]
r[w_,x_,y_,z_]:= -or[y,z,x,w]/;
  SameQ[w,Sort[{x,y,z,w}][[4]]]

r[x1_,x2_,x3_,x4_,x5_]:= rp[x1,x2,x3,x4,x5]/;
  SameQ[x5,Sort[{x1,x2,x3,x4,x5}][[5]]]
r[x1_,x2_,x3_,x5_,x4_]:= -rp[x1,x2,x3,x4,x5]/;
  SameQ[x5,Sort[{x1,x2,x3,x4,x5}][[5]]]
r[x1_,x2_,x5_,x3_,x4_]:= rp[x1,x3,x4,x2,x5]/;
  SameQ[x5,Sort[{x1,x2,x3,x4,x5}][[5]]]
r[x1_,x5_,x2_,x3_,x4_]:= -rp[x1,x3,x4,x2,x5]/;
  SameQ[x5,Sort[{x1,x2,x3,x4,x5}][[5]]]
r[x5_,x1_,x2_,x3_,x4_]:=
  -rp[x1,x3,x4,x2,x5] + rp[x2,x3,x4,x1,x5]/;
  SameQ[x5,Sort[{x1,x2,x3,x4,x5}][[5]]]

subst11="{
(* r(4)^2, r(4)^3 *)
  or[x_,y_,z_,w_]^2->csq,
(* The arguments of the 2nd or[] are
   x_,y_,z_,w_. *)
  or[x_,z_,y_,w_]*or[]-> csq/2,
  or[y_,x_,z_,w_]*or[]-> -csq,
  or[y_,z_,x_,w_]*or[]-> -csq/2,
  or[z_,x_,y_,w_]*or[]-> -csq/2,
  or[z_,y_,x_,w_]*or[]-> csq/2,

  or[x1_,y_,x2_,w_]*or[x3_,x4_,y_,w_]->
    or[x1,x2,y,w]*or[x3,x4,y,w]/2,
  or[y_,x1_,x2_,w_]*or[x3_,x4_,y_,w_]->
    -or[x1,x2,y,w]*or[x3,x4,y,w]/2,
  or[x1_,y_,x2_,w_]*or[x3_,y_,x4_,w_]->
    u[x1,x2]*u[x3,x4],
  or[y_,x1_,x2_,w_]*or[x3_,y_,x4_,w_]->
    -u[x1,x2]*u[x3,x4],
  or[y_,x1_,x2_,w_]*or[y_,x3_,x4_,w_]->
    u[x1,x2]*u[x3,x4],
(* r(5)^2 *)
  rp[x1_,x2_,x3_,x4_,x5_]^2->dcsq2,
(* The arguments of the 1st rp[] are
   x1_,x2_,x3_,x4_,x5_. *)
  rp[]*rp[x1_,x2_,x4_,x3_,x5_]-> dcsq2/2,
  rp[]*rp[x1_,x3_,x2_,x4_,x5_]-> -dcsq2,
  rp[]*rp[x1_,x3_,x4_,x2_,x5_]-> -dcsq2/2,
  rp[]*rp[x1_,x4_,x2_,x3_,x5_]-> -dcsq2/2,
  rp[]*rp[x1_,x4_,x3_,x2_,x5_]-> dcsq2/2,
```

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  rp[]*rp[x2_,x1_,x3_,x4_,x5_]-> dcsq2/2,
  rp[]*rp[x2_,x1_,x4_,x3_,x5_]-> dcsq2/4,
  rp[]*rp[x2_,x3_,x1_,x4_,x5_]-> -dcsq2/2,
  rp[]*rp[x2_,x4_,x1_,x3_,x5_]-> -dcsq2/4,
  rp[]*rp[x3_,x1_,x2_,x4_,x5_]-> -dcsq2/2,
  rp[]*rp[x3_,x1_,x4_,x2_,x5_]-> -dcsq2/4,
  rp[]*rp[x3_,x2_,x1_,x4_,x5_]-> dcsq2/2,
  rp[]*rp[x3_,x4_,x1_,x2_,x5_]-> dcsq2/4,

  rp[]*rp[x2_,x3_,x4_,x1_,x5_]-> -dcsq2/4,
  rp[]*rp[x2_,x4_,x3_,x1_,x5_]-> dcsq2/4,
  rp[]*rp[x3_,x2_,x4_,x1_,x5_]-> dcsq2/4,
  rp[]*rp[x3_,x4_,x2_,x1_,x5_]-> -dcsq2/4,
  rp[]*rp[x4_,x1_,x2_,x3_,x5_]-> -dcsq2/4,
  rp[]*rp[x4_,x1_,x3_,x2_,x5_]-> dcsq2/4,
  rp[]*rp[x4_,x2_,x1_,x3_,x5_]-> dcsq2/4,
  rp[]*rp[x4_,x3_,x1_,x2_,x5_]-> -dcsq2/4,

  rp[]*rp[x4_,x2_,x3_,x1_,x5_]-> dcsq2/2,
  rp[]*rp[x4_,x3_,x2_,x1_,x5_]-> -dcsq2/2}";
Do[h[j,counter+1]=Expand[h[j,counter]]//.
  ToExpression[subst11],{j,1,4}]
counter=counter+1

subst12="{
  or[x1_,x2_,x3_,x4_]*or[x1_,x2_,y_,z_]*
    or[x3_,x4_,y_,z_]->ccu1,
  or[x1_,x2_,x3_,x4_]*or[x2_,x1_,y_,z_]*
    or[x3_,x4_,y_,z_]->-ccu1,
  or[x1_,x2_,x3_,x4_]*or[x1_,x2_,y_,z_]*
    or[x4_,x3_,y_,z_]->-ccu1,
  or[x1_,x2_,x3_,x4_]*or[x2_,x1_,y_,z_]*
    or[x4_,x3_,y_,z_]->ccu1,

  or[x1_,x2_,x3_,x4_]*or[x1_,x3_,y_,z_]*
    or[x2_,x4_,y_,z_]->ccu1/2,
  or[x1_,x2_,x3_,x4_]*or[x3_,x1_,y_,z_]*
    or[x2_,x4_,y_,z_]->-ccu1/2,
  or[x1_,x2_,x3_,x4_]*or[x1_,x3_,y_,z_]*
    or[x4_,x2_,y_,z_]->-ccu1/2,
  or[x1_,x2_,x3_,x4_]*or[x3_,x1_,y_,z_]*
    or[x4_,x2_,y_,z_]->ccu1/2,
(* Similar expressions where x3 and x4 in the
  second and the third or[] are exchanged
  follow. *)
(* The arguments of or[] are
   x1_,x2_,x3_,x4_. *)
  or[]*u[x1_,x2_]*u[x3_,x4_]-> ccu1/4,
  or[]*u[x2_,x1_]*u[x3_,x4_]-> -ccu1/4,
  or[]*u[x1_,x2_]*u[x4_,x3_]-> -ccu1/4,
  or[]*u[x2_,x1_]*u[x4_,x3_]-> ccu1/4,

  or[]*u[x1_,x3_]*u[x2_,x4_]-> ccu2,
  or[]*u[x3_,x1_]*u[x2_,x4_]-> ccu2-ccu1/4,
  or[]*u[x1_,x3_]*u[x4_,x2_]-> ccu2-ccu1/4,
  or[]*u[x3_,x1_]*u[x4_,x2_]-> ccu2,
(* Similar expressions where x3 and x4 in u[]
  are exchanged follow. *)
  aaa->-cddc/4-rcsq/2-ccu1/4-ccu2}";

Do[h[j,counter+1]=Expand[h[j,counter]]//.
ToExpression[subst12],{j,1,4}]
counter=counter+1

so1= Expand[h[1,counter]]
so2= Expand[h[2,counter]]
so3= Expand[h[3,counter]]
so4= Expand[h[4,counter]]

Save["s8cont_r.txt",so1,so2,so3,so4]
```

References and Notes

[1] K. Seo, *Bull. Gifu City Wom. Col.*, No.50, 2001, p.63; *ibid*, No.51, 2002, p.89; *ibid*, No.52, 2003, p.73.

[2] In Ref.1, the overall factors of the trace anomaly are miscounted. The factors should be $\frac{1}{4\pi}$ and $\frac{1}{16\pi^2}$ in two and four space-time dimensions, respectively, instead of $\frac{1}{2\pi}$ and $\frac{1}{4\pi^2}$.

[3] See for example, M.B. Green, J.H. Schwartz and E. Witten, *Superstring theory*, Vol.1 and 2, (Cambridge University Press), 1987.

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[8] P.B. Gilkey, *J. Differential Geometry*, Vol.10, 1975, p.601.

[9] In order to compare our result with Gilkey's one, E in Ref.8 should be substituted by $R/4$. R_{ijij} and R_{ijik} in Ref.8 should be read as the scalar curvature R and the Ricci tensor R_{jk}, respectively.

[10] I.G. Avramidi, *Nucl. Phys. B*, Vol.355, 1991, p.712, Erratum-ibid. Vol.509, 1998, p.557; I.G. Avramidi, *Nucl. Phys. B*, Vol.355, 1991, p.712.

（提出期日　平成 15 年 12 月 10 日）